

BERLINE-VERGNE VALUATION AND GENERALIZED PERMUTOHEDRA

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ABSTRACT. Generalizing a conjecture by De Loera et al., we conjecture that all the integral generalized permutohedra have positive Ehrhart coefficients. Berline-Vergne construct a valuation that assign values to faces of polytopes, which provides a way to write Ehrhart coefficients of a polytope as positive sums of these values. Based on empirical results, we conjecture Berline-Vergne's valuation is always positive on regular permutohedra, which implies our first conjecture.

This article proves that our conjecture on Berline-Vergne's valuation is true for dimension up to 6, and is true if we restrict to faces of codimension up to 3. We also give two equivalent statements to this conjecture in terms of mixed valuations and Todd class, respectively. In addition to investigating the positivity conjectures, we study the Berline-Vergne's valuation, and show that it is the unique construction for McMullen's formula (used to describe number of lattice points in a polytope) under certain symmetry constraints.

1. INTRODUCTION

A *lattice point* is a point in \mathbb{Z}^D . Given any bounded set $S \subseteq \mathbb{R}^D$, we let

$$\text{Lat}(S) := |S \cap \mathbb{Z}^D|$$

be the number of lattice points in S . Given a polytope P in \mathbb{R}^D , a natural enumerative problem is to compute $\text{Lat}(P)$. In this paper, we will focus on *integral polytopes*, i.e., polytopes whose vertices are all lattice points, and generalized permutohedra – a special family of polytopes.

1.1. Motivation: Ehrhart positivity for generalized permutohedra. One approach to study the question of computing $\text{Lat}(P)$ for an integral polytope P is to consider a more general counting problem: For any nonnegative integer t , let $tP := \{t\mathbf{x} \mid \mathbf{x} \in P\}$ be the t -th *dilation* of P , and then consider the function

$$i(P, t) := \text{Lat}(tP)$$

that counts the number of lattice points in tP . It is a classic result that $i(P, t)$ is a polynomial in t . More precisely:

Theorem 1.1 (Ehrhart [6]). *There exists a polynomial $f(x)$ such that $f(t) = i(P, t)$ for any $t \in \mathbb{Z}_{\geq 0}$. Moreover, the degree of $f(x)$ is equal to the dimension of P .*

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We call the function $i(P, t)$ the *Ehrhart polynomial* of P . A few coefficients of $i(P, t)$ are well understood: the leading coefficient is equal to the normalized volume of P , the second coefficient is one half of the sum of the normalized volumes of facets, and the constant term is always 1. However, although formulas are derived for the other coefficients, they are quite complicated. One notices that the leading, second and last coefficients of the Ehrhart polynomial of any integral polytope are always positive; but it is not true for the rest of the coefficients for general polytopes. We say a polytope has *Ehrhart positivity* or is *Ehrhart positive* if it has positive Ehrhart coefficients.

There are few families of polytopes known to be Ehrhart positive. Zonotopes, in particular the regular permutohedra, are Ehrhart positive [15, Theorem 2.2]. Cyclic polytopes also have this property. Their Ehrhart coefficients are given by the volumes of certain projections of the original polytope [9]. Stanley-Pitman polytopes are defined in [17] where a formula for the Ehrhart polynomial is given and from which Ehrhart positivity follows. Recently in [5] De Loera, Haws, and Koeppel study the case of matroid base polytopes and conjecture they are Ehrhart positive. Both Stanley-Pitman polytopes and matroid base polytopes fit into a bigger family: generalized permutohedra.

In [14] Postnikov defines generalized permutohedra as polytopes obtained by moving the vertices of a usual permutohedron while keeping the same edge directions. That's what he calls a generalized permutohedron of type z . He also considers a strictly smaller family, type y , consisting of sums of dilated simplices. He describes the Ehrhart polynomial for the type y family in [14, Theorem 11.3], from which Ehrhart positivity follows. The type y family includes the Stanley-Pitman polytopes, associahedra, cyclohedra, and more (see [14, Section 8]), but fails to contain matroid base polytopes, which are type z generalized permutohedra [1, Proposition 2.4].

We give the following conjecture:

Conjecture 1.2. *Integral generalized permutohedra are Ehrhart positive.*

Note that since generalized permutohedra contain the family of matroid base polytopes, our conjecture is a generalization of the conjecture on Ehrhart positivity of matroid base polytopes given in [5] by De Loera et al.

Instead of studying the above conjecture directly, we will reduce it to another conjecture which only concerns regular permutohedra, a smaller family of polytopes.

1.2. McMullen's formula and α -positivity. In 1975 Danilov asked if it is possible to assign values r_σ to cones σ such that

$$(1.1) \quad \text{Td}(X(\Delta)) = \sum_{\sigma \in \Delta} r_\sigma [V(\sigma)],$$

where Δ is a complete fan, $X(\Delta)$ is the corresponding toric variety, and $V(\sigma)$ is the closed subvariety associated with σ . Notice the special feature of Formula (1.1): the value of r_σ only depends on the cone σ , but not on the fan Δ . We call this the *Danilov condition*.

If such an expression exists, then the following *McMullen's formula* holds for any integral polytope P :

$$(1.2) \quad \text{Lat}(P) = \sum_{\substack{F: \text{ a face of } P \\ 2}} \alpha(F, P) \text{ nvol}(F),$$

where $\alpha(F, P)$ is set to be r_σ where σ is the normal cone of P at F , and $\text{nvol}(F)$ is the normalized volume of F .

In fact, one can ask directly the existence of McMullen's formula (independently from Danilov's question). More specifically, one can ask whether there are ways to assign values to cones such that McMullen's formula holds if $\alpha(F, P)$ only depends on the normal cone of P at F . We will discuss the implication (1.1) \Rightarrow (1.2) in Section 7. But for most the paper, we focus on McMullen's formula.

McMullen [10] was the first to confirm the existence of Formula (1.2) in a non constructive way (which was the reason we call this formula McMullen's formula). Morelli [11] supplied the first explicit way to choose r_σ answering Danilov's question. Pommersheim and Thomas [12] gave a canonical construction of r_σ based on choices of flags. As we discussed above, both of these two constructions naturally give a way to construct α for McMullen's formula (1.2). Berline and Vergne [4] were able to construct such α in a computable way. One immediate consequence of the existence of Formula (1.2) is that if $\alpha(F, P)$ is positive for each face F of P , then Ehrhart positivity follows. (See Theorem 3.1 and Lemma 3.2.) Hence, it is natural to say that a polytope P has α -positivity or is α -positive if all α 's associated to P are positive.

Although there are different constructions for $\alpha(F, P)$, Berline-Vergne's construction has certain nice properties that are good for our purpose, and thus we will use their construction in our paper. We refer to their construction for $\alpha(F, P)$ as the *BV- α -valuation*. To be more precise, we will use the terminologies *BV- α -positivity* and *BV- α -positive* to indicate the α 's we use are from the BV- α -valuation.

At present, the explicit computation of the BV- α -valuation is a recursive, complicated process, but we carry it out in the special example of regular permutohedra of small dimensions, whose symmetry simplifies the computations. Based on our empirical results, we conjecture the following:

Conjecture 1.3. *Every regular permutohedron is BV- α -positive.*

One important property of the BV- α -valuation enables us to reduce the problem of proving the Ehrhart positivity of all generalized permutohedra to proving the positivity of all the α 's arising from the regular permutohedra.

Theorem 1.4. *Conjecture 1.3 implies Conjecture 1.2.*

Therefore, we focus on proving Conjecture 1.3 instead. In this paper, we provide partial progress on proving Conjecture 1.3 (and thus Conjecture 1.2), as well as present equivalent statements to Conjecture 1.3 in terms of mixed valuations and Todd class respectively.

Organization of the paper. In Section 2, we give basic definitions and review background results that are relevant to our paper. In Section 3, we give details of the BV- α -valuation, discuss consequences of the properties of this construction. In particular, we complete the proof of Theorem 1.4.

In Section 4, assuming the α function in McMullen's formula (1.2) is symmetric about the coordinates (a property of the BV- α -valuation), we derive a combinatorial formula for $\text{Lat}(\text{Perm}(\mathbf{v}))$ indexed by subsets of $[n]$, where $\text{Perm}(\mathbf{v})$ is a *generic permutohedron*, which belongs to a family of generalized permutohedra containing the regular permutohedra. In Section 5, we carry out direct computation to find values of BV- α -valuations for regular permutohedra, and verify that Conjecture 1.3 is true for dimension up to 6. We are also

able to verify positivity of $\alpha(F, \Pi_n)$ for faces F of codimension 2 or 3. These results provide evidences to Conjecture 1.3.

In addition to investigating positivity of the α 's, there are other questions one can ask about these constructions. Although there are different constructions for α , one still wonders whether under certain constraints, the construction is unique. In Section 6, using the combinatorial formula we derived in Section 4 and theories of mixed valuations, we show that the α -values arising from the regular permutohedron is unique as long as we assume α is symmetric about the coordinates. As a consequence of our techniques, we give an equivalent statement to Conjecture 1.3 in terms of mixed valuations (Corollary 6.6).

In Section 7, we quickly review the connection with toric varieties as in [8, Section 5.3]. Analogously to the treatment of mixed volumes in [8, Section 5.4], we can relate the more general mixed valuations to some intersection theoretic quantities. With these arguments we can give one more equivalence of our main Conjecture 1.3. It relates to the positivity of the Todd class of the permutohedral variety (in some) expression in terms of the torus invariant cycles.

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2. BACKGROUND

In this section and the next section, we assume the ambient space is \mathbb{R}^D , and \mathbb{Z}^D is the *lattice* in \mathbb{R}^D . For any D -vector α , $\alpha_i(\cdot)$ is the linear function that maps $\mathbf{x} \in \mathbb{R}^D$ to the scalar product of α and \mathbf{x} . Since we can consider $\alpha(\cdot)$ as a point in the dual space $(\mathbb{R}^D)^*$ of \mathbb{R}^D , we will use the notation α (or any bold letter) to denote both the D -vector and the linear function.

We assume familiarity with basic definitions of polyhedra and polytopes as presented in [?, ?], and only review terminologies and setups that are relevant to us.

A *polyhedron* is the set of points defined by a linear system of equalities and inequalities

$$(2.1) \quad \begin{aligned} \alpha_i(\mathbf{x}) &= w_i, \quad \forall 1 \leq i \leq m_1, \\ \beta_i(\mathbf{x}) &\leq z_i, \quad \forall 1 \leq i \leq m_2. \end{aligned}$$

For simplicity, we let A be the $m_1 \times D$ matrix whose row vectors are α_i 's, B the $m_2 \times D$ matrix whose row vectors are β_i 's, $\mathbf{w} = (w_1, \dots, w_{m_1})^T$, and $\mathbf{z} = (z_1, \dots, z_{m_2})^T$, so the above linear system can be represented as

$$A\mathbf{x} = \mathbf{w}, \quad B\mathbf{x} \leq \mathbf{z}.$$

A *polytope* is a bounded polyhedron. (A polytope can also be defined as the convex hull of a finite set of points.)

For any polyhedron P , we use $\text{vert}(P)$ to denote the vertex set of P . An *integral* polyhedron is a polyhedron whose vertices are all lattice points, i.e., points with integer coordinates.

Let V be a subspace of \mathbb{R}^D , and $\Lambda := V \cap \mathbb{Z}^D$ the lattice in V . For any polytope P that is lying in an affine space that is a translation of V , we define the *volume of P normalized to the lattice Λ* to be the integral

$$\text{Vol}_\Lambda(P) := \int_P 1 \, d\Lambda,$$

where $d\Lambda$ is the canonical Lebesgue measure defined by the lattice Λ . In the case where $\dim P = \dim \Lambda$, we get the *normalized volume* of P , denoted by $\text{nvol}(P)$.

2.1. Cones and fans. A (*polyhedral*) *cone* is the set of all nonnegative linear combinations of a finite set of vectors. A *shifted cone* is a set of points in the form of $C + \mathbf{x}$ where C a cone and \mathbf{x} is a point. A shifted cone is *pointed* if it does not contain a line. A shifted cone $C + \mathbf{x}$ is *rational* if the cone C is generated by vectors with rational coordinates.

Definition 2.1. Suppose P is a polyhedron and F is a face. The *tangent cone* of F at P is:

$$\text{tcone}(F, P) = \{F + \mathbf{u} : F + \delta\mathbf{u} \in P \text{ for sufficiently small } \delta\}.$$

The *feasible cone* of F at P is:

$$\text{fcone}(F, P) = \{\mathbf{u} : F + \delta\mathbf{u} \in P \text{ for sufficiently small } \delta\}$$

Note that $\text{tcone}(F, P)$ is a shifted cone, but not necessarily a cone, when $\text{fcone}(F, P)$ is always a cone.

In order to always work with pointed cones, we also define

$$\text{tcone}^p(F, P) = \text{tcone}(F, P)/L \quad \text{and} \quad \text{fcone}^p(F, P) = \text{fcone}(F, P)/L$$

where L is the affine space spanned by F . Then $\text{tcone}^p(F, P)$ and $\text{fcone}^p(F, P)$ are pointed (shifted) cones with dimension $\dim P - \dim F$.

Definition 2.2. Suppose V is a subspace of \mathbb{R}^D . Let $K \subseteq V$ be a cone. The *polar cone* of K with respect to V is the cone

$$K_V^\circ = \{\mathbf{y} \in V^* \mid \mathbf{y}(\mathbf{x}) \leq 0, \forall \mathbf{x} \in K\}.$$

In the situation where K is full-dimensional in V , we will omit the subscript V and the words “with respect to V ”.

Definition 2.3. Suppose V is a subspace of \mathbb{R}^D and $P \subset V + \mathbf{y}$ for some $\mathbf{y} \in \mathbb{R}^D$. Given any face F of P , the *normal cone* of P at F with respect to V is

$$\text{ncone}_V(F, P) := \{\mathbf{u} \in V^* : \mathbf{u}(\mathbf{p}_1) \geq \mathbf{u}(\mathbf{p}_2), \quad \forall \mathbf{p}_1 \in F, \quad \forall \mathbf{p}_2 \in P\}.$$

Therefore, $\text{ncone}_V(F, P)$ is the collection of linear functions \mathbf{u} in V^* such that \mathbf{u} attains maximum value at F over all points in P .

The *normal fan* $\Sigma_V(P)$ of P with respect to V is the collection of all normal cones of P .

In the situation where the affine span of P is $V + \mathbf{y}$, i.e., $\dim(P) = \dim(V)$, we will omit the subscript V and the words “with respect to V ”.

We have the following easy results for normal cones which will be useful for our paper.

Lemma 2.4. *Let L be the shift of the affine span of F to the origin. Then $\text{ncone}_V(F, P)$ spans the orthogonal complement of L with respect to V . Hence,*

$$(2.2) \quad \text{ncone}_V(F, P) = \dim V - \dim F.$$

Furthermore, the pointed feasible cone of P at F and the normal cone of P at F are polar to one another in the following sense:

$$(2.3) \quad (\text{fcone}^p(F, P))_{V/L}^\circ = \text{ncone}_V(F, P) \text{ and } (\text{ncone}_V(F, P))_{V/L}^\circ = \text{fcone}^p(F, P).$$

2.2. Generalized permutohedra. We introduce *generalized permutohedra*, the main family of polytopes we study in this paper. In this part and any later part that is related to generalized permutohedra, we assume $D = n + 1$, i.e., the ambient is \mathbb{R}^{n+1} . First, we present the *usual permutohedron* as the convex hull of a finite number of points.

Definition 2.5. Given a point $\mathbf{v} = (v_1, v_2, \dots, v_{n+1}) \in \mathbb{R}^{n+1}$, we construct the *usual permutohedron*

$$\text{Perm}(\mathbf{v}) = \text{Perm}(v_1, v_2, \dots, v_{n+1}) := \text{conv} \left(v_{\sigma(1)}, v_{\sigma(2)}, \dots, v_{\sigma(n+1)} \right) : \quad \sigma \in \mathfrak{S}_{n+1}$$

In particular, if $\mathbf{x} = (1, 2, \dots, n + 1)$, we obtain the *regular permutohedron*, denoted by Π_n ,

$$\Pi_n := \text{Perm}(1, 2, \dots, n + 1).$$

Note that as long as there are two different entries in \mathbf{v} we have $\dim(\text{Perm}(\mathbf{v})) = n$.

The *generalized permutohedra* is originally introduced by Postnikov [14, Definition 6.1] as polytopes obtained from usual permutohedra by moving vertices while preserving all edge directions. In [13], Postnikov, Reiner, and Williams give several equivalent definitions, one of which uses concepts of normal fans.

Definition 2.6. Let V be the subspace of \mathbb{R}^{n+1} defined by $x_1 + x_2 + \dots + x_{n+1} = 0$. The *braid arrangement fan*, denoted by \mathfrak{B}_n , is the complete fan in V given by the hyperplanes

$$x_i - x_j = 0 \quad \text{for all } i \neq j.$$

Proposition 2.7 (Proposition 3.2 of [13]). *Let V be the subspace of \mathbb{R}^{n+1} defined by $x_1 + x_2 + \dots + x_{n+1} = 0$. A polytope P in \mathbb{R}^{n+1} is a generalized permutohedron if and only if its normal fan $\Sigma_V(P)$ with respect to V is refined by the braid arrangement fan \mathfrak{B}_n .*

It follows from [14, Proposition 2.6] that as long as $\mathbf{v} = (v_1, v_2, \dots, v_{n+1})$ has distinct coordinates, the associated usual permutohedron $\text{Perm}(\mathbf{v})$ has the braid arrangement \mathfrak{B}_n as its normal fan. We call $\text{Perm}(\mathbf{v})$ with \mathbf{v} of distinct coordinates a *generic permutohedron*. In particular, the regular permutohedron Π_n is a generic permutohedron.

2.3. Indicator functions and algebra of polyhedra. For a set $S \subseteq \mathbb{R}^D$, the indicator function $[S] : \mathbb{R}^D \rightarrow \mathbb{R}$ of S is defined as

$$[S](x) = \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{if } x \notin S. \end{cases}$$

Let V be a subspace of \mathbb{R}^D . The *algebra of polyhedra*, denoted by $\mathcal{P}(V)$, is the vector space defined as the span of the indicator functions of all polyhedra in V . We similarly define $\mathcal{P}_b(V)$ as the *algebra of polytopes*. For any $\mathbf{x} \in \mathbb{R}^D$, the *algebra of shifted cones at \mathbf{x}* , denoted by $\mathcal{C}_{\mathbf{x}}(V)$, is the vector space defined as the span of the indicator functions of all shifted cones that are in the form of $C + \mathbf{x}$ for some cone C .

A linear transformation $\phi : \mathcal{P}(V), \mathcal{P}_b(V), \mathcal{C}_{\mathbf{x}}(V) \rightarrow W$, where W is a vector space, is a *valuation*.

Both volume $\text{Vol}_{\Lambda}(\cdot)$ and number of lattice points $\text{Lat}(\cdot)$ are valuations on the algebra of polytopes. However, normalized volume $\text{nvol}(\cdot)$ is not a valuation.

Below is an important result on indicator functions of feasible cones at vertices.

Theorem 2.8 (Theorem 6.6 of [2]). *Suppose P is a nonempty polytope. Then*

$$[0] \equiv \sum_{v: \text{ a vertex of } P} [\text{fcone}^P(v, P)] \quad \text{modulo polyhedra with lines}$$

2.4. Mixed valuations. Let Λ be a sublattice of \mathbb{Z}^D and V is the span of Λ .

A valuation is a Λ -valuation if it is invariant under Λ -translation. We say a valuation ϕ is *homogeneous of degree d* if $\phi([tP]) = t^d \phi([P])$ for any integral polytope P and $t \in \mathbb{Z}_{\geq 0}$. It's clear that Vol_Λ is homogeneous of degree $\dim \Lambda$, but Lat is not homogenous.

The following important theorem by McMullen is a special case of [10, Theorem 6].

Theorem 2.9. *Suppose ϕ is a homogeneous Λ -valuation on $\mathcal{P}_b(V)$ of degree d . Then there exists a function \mathcal{M} which takes d integral polytopes as inputs such that*

$$(2.4) \quad \phi(t_1 P_1 + t_2 P_2 + \cdots + t_k P_k) = \sum_{j_1, \dots, j_d \in [d]} \mathcal{M}(P_{j_1}, P_{j_2}, \dots, P_{j_d}) t_{j_1} \cdots t_{j_d},$$

for any $k \in \mathbb{Z}_{>0}$, any integral polytopes $P_1, \dots, P_k \subset V$ and $t_1, \dots, t_k \in \mathbb{Z}_{\geq 0}$.

The following definition and lemma are stated in [7, Section 3 of Chapter IV] for the volume valuation (which is a homogeneous valuation). We give the general forms here.

Definition 2.10. Let ϕ and \mathcal{M} be as in Theorem 2.9. We define another function $\mathcal{M}\phi$ that takes d integral polytopes as inputs as an average of the function \mathcal{M} :

$$\mathcal{M}\phi(P_1, \dots, P_d) := \frac{1}{d!} \sum_{\sigma \in \mathfrak{S}_d} \mathcal{M}(P_{\sigma(1)}, \dots, P_{\sigma(d)}).$$

It is easy to see that $\mathcal{M}\phi$ is uniquely chosen for each ϕ , and (2.4) still holds for $\mathcal{M}\phi$:

$$(2.5) \quad \phi(t_1 P_1 + t_2 P_2 + \cdots + t_k P_k) = \sum_{j_1, \dots, j_d \in [d]} \mathcal{M}\phi(P_{j_1}, P_{j_2}, \dots, P_{j_d}) t_{j_1} \cdots t_{j_d},$$

We call $\mathcal{M}\phi$ the *mixed valuation* of ϕ .

The lemma below gives two properties of the mixed valuation $\mathcal{M}\phi$.

Lemma 2.11. (i) *For any integral polytopes P_1, \dots, P_d , and any permutation $\sigma \in \mathfrak{S}_d$, we have*

$$(2.6) \quad \mathcal{M}\phi(P_1, \dots, P_d) = \mathcal{M}\phi(P_{\sigma(1)}, \dots, P_{\sigma(d)}).$$

(ii) *The function $\mathcal{M}\phi$ is a multi-linear function, that is, it is linear in each component.*

Proof. (i) follows directly from the definition of $\mathcal{M}\phi$. For (ii), we will just prove $\mathcal{M}\phi$ is linear in the first component, that is, to show for any integral polytopes $P_1, P'_1, P_2, P_3, \dots, P_n$ and nonnegative integers s_1, s'_1 , we have

$$(2.7) \quad \mathcal{M}\phi(s_1 P_1 + s'_1 P'_1, P_2, \dots, P_d) = s_1 \mathcal{M}\phi(P_1, P_2, \dots, P_d) + s'_1 \mathcal{M}\phi(P'_1, P_2, \dots, P_d),$$

We apply (2.5) to both sides of the following equality:

$$\phi(t_1(s_1 P_1 + s'_1 P'_1) + t_2 P_2 + \cdots + t_d P_d) = \phi((t_1 s_1) P_1 + (t_1 s'_1) P'_1 + t_2 P_2 + \cdots + t_d P_d).$$

Consider s_1 and s'_1 as fixed numbers. Then each side gives a homogeneous polynomial in t_1, t_2, \dots, t_d . Since these two homogeneous polynomials agree on all $t_1, t_2, \dots, t_n \in \mathbb{Z}_{\geq 0}$, we conclude that they are exactly the same polynomials, and thus their coefficients agree. Then (2.7) follows from (2.6) and comparing the coefficients of $t_1 t_2 \dots t_n$. \square

Apply the above results to volume valuation, a homogeneous valuation, we obtain the following:

Theorem 2.12 (Theorem 3.2 of [7]). *Suppose P_1, \dots, P_k are integral polytopes with $\dim(P_1 + \dots + P_k) = d$. Let Λ be the d -dimensional lattice $\text{span}(P_1 + \dots + P_k) \cap \mathbb{Z}^D$. Then*

$$\text{Vol}_\Lambda(t_1 P_1 + t_2 P_2 + \dots + t_k P_k) = \sum_{j_1, \dots, j_d=1}^d \mathcal{M}\text{Vol}_\Lambda(P_{j_1}, P_{j_2}, \dots, P_{j_d}) t_{j_1} \dots t_{j_d}$$

where the sum is carried out independently over the j_i . The function $\mathcal{M}\text{Vol}_\Lambda(P_{j_1}, P_{j_2}, \dots, P_{j_d})$ is called the mixed volume of $P_{j_1}, P_{j_2}, \dots, P_{j_d}$.

Furthermore we have the following properties:

Theorem 2.13 (Theorem 4.13 of [7]). *Let P_1, \dots, P_d be integral polytopes. Then,*

- (1) $\mathcal{M}\text{Vol}_\Lambda(P_1, \dots, P_d) \geq 0$
- (2) $\mathcal{M}\text{Vol}_\Lambda(P_1, \dots, P_d) > 0$ if and only if each P_i contains a line segment $I_i = [a_i, b_i]$ such that $b_1 - a_1, \dots, b_d - a_d$ are linearly independent.

The lattice point, or counting, valuation Lat is not homogeneous. However it can be decomposed into homogeneous parts.

Theorem 2.14 (Theorem 5 of [10]). *Suppose V is d -dimensional. Then we can decompose the valuation Lat as*

$$\text{Lat} = \text{Lat}^d + \dots + \text{Lat}^1 + \text{Lat}^0$$

where Lat^r is homogenous of degree r .

This decomposition corresponds to the coefficients of the Ehrhart polynomial, in particular Lat^d corresponds to the volume valuation Vol_Λ , where $\Lambda = V \cap \mathbb{Z}^D$. Applying Theorem 2.9 and Lemma 2.11 to each homogeneous function Lat^r gives us the following result:

Theorem 2.15. *Suppose P_1, \dots, P_k are integral polytopes with $\dim(P_1 + \dots + P_k) = d$. Then*

$$\text{Lat}(t_1 P_1 + t_2 P_2 + \dots + t_k P_k) = \sum_{e=0}^d \sum_{j_1, \dots, j_e=1}^e \mathcal{M}\text{Lat}^e(P_{j_1}, P_{j_2}, \dots, P_{j_e}) t_{j_1} \dots t_{j_e}.$$

We cannot expect $\mathcal{M}\text{Lat}^r$ or any other mixed valuation $\mathcal{M}\phi$ to be nonnegative in general. However we have a way to compute them.

Theorem 2.16. *Suppose ϕ is a homogeneous Λ -valuation on $\mathcal{P}_b(V)$ of degree d . For any integral polytopes $P_1, P_2, \dots, P_d \subset V$, we have*

$$d! \mathcal{M}\phi(P_1, P_2, \dots, P_d) = \sum_{J \subseteq [d]} (-1)^{d-|J|} \phi \left(\sum_{j \in J} P_j \right)$$

Proof. We define two functions f and g on the boolean algebra of order d . (See [16, Chapter 3].) For any subset $T \subseteq [d]$, let

$$f(T) := \phi \left(\sum_{i \in T} P_i \right) \quad \text{and} \quad g(T) := \sum_{\substack{j_1, \dots, j_d \in [d] \\ \cup_{i=1}^d \{j_i\} = T}} \mathcal{M}\phi(P_{j_1}, P_{j_2}, \dots, P_{j_d}).$$

Apply (2.5) with $t_i = 1$ to $f(T)$, one sees that $f(T) = \sum_{S \subseteq T} g(S)$. Therefore, by Mobius inversion, we get

$$g(T) = \sum_{S \subseteq T} (-1)^{|T|-|S|} f(S).$$

Then the theorem follows from evaluate the above equality at $T = [d]$. \square

3. McMULLEN'S FORMULA AND THE BV- α -VALUATION

Recall in the introduction, we've discussed the question of the existence of the following *McMullen's formula*

$$(3.1) \quad \text{Lat}(P) = \sum_{F: \text{ a face of } P} \alpha(F, P) \text{ nvol}(F)$$

where $\alpha(F, P)$ depends only on the normal cone of P at F .

One immediate consequence of the existence of McMullen's formula (3.1) is that it provides another way to prove Ehrhart's theorem. Moreover, it gives a description of each Ehrhart coefficient. We state the following modified version of Theorem 1.1.

Theorem 3.1. *For an integral polytope $P \subset \mathbb{Z}^D$ and any $t \in \mathbb{Z}_{\geq 0}$, the function*

$$i(P, t) = \text{Lat}(tP) = |tP \cap \mathbb{Z}^n|$$

is a polynomial in t of degree $\dim P$. Furthermore, the coefficient of t^k in $i(P, t)$ is

$$(3.2) \quad \sum_{F: \text{ a } k\text{-dim face of } P} \alpha(F, P) \text{ nvol}(F).$$

Proof. When we dilate the polytope P by a factor of t , each face F of P becomes tF , a face of tP . It is clear that the normal cone does not change. Hence, applying McMullen's formula to tP , we get

$$i(P, t) = \sum_F \alpha(tF, tP) \text{ nvol}(tF) = \sum_F \alpha(F, P) \text{ nvol}(F) t^{\dim F}.$$

Then our conclusion follows. \square

Formula (3.2) for the coefficients of the Ehrhart polynomial $i(P, t)$ gives a sufficient condition for Ehrhart positivity.

Lemma 3.2. *Let P be an integral polytope. For a fixed k , if $\alpha(F, P)$ is positive for any k -dim face of P , then the coefficient of t^k in the Ehrhart polynomial $i(P, t)$ of P is positive.*

Hence, if for every face F of P , we have $\alpha(F, P) > 0$, then P is Ehrhart positive.

As discussed in the introduction, different constructions of $\alpha(F, P)$ were given in the literature. In our paper, we will use Berline-Vergen's construction, which we refer to as the *BV- α -valuation*.

3.1. Berline-Vergne's construction. Berline and Vergne construct in [4] a function $\Psi([C], \Lambda)$ on indicator functions of rational shifted cones C with respect to a lattice Λ , where Λ is a quotient of the lattice $V \cap \mathbb{Z}^D$ and C is inside the affine span of Λ . Then they show Ψ has the following properties:

- (P1) Let V be the affine span of a lattice Λ . Then $\Psi(\cdot, \Lambda)$ is a valuation on $\mathcal{C}_{\mathbf{x}}(V)$ for any $\mathbf{x} \in V$. (Recall that $\mathcal{C}_{\mathbf{x}}(V)$ is the algebra of shifted cones at \mathbf{x} .)
- (P2) McMullen's formula (3.1) holds for rational polytopes if we set

$$(3.3) \quad \alpha(F, P) := \Psi([\text{tcone}^p(F, P)], (V \cap \mathbb{Z}^D)/L),$$

where V and L are the affine spaces spanned by P and F respectively.

- (P3) If a cone C contains a line, then $\Psi([C], \Lambda) = 0$.
- (P4) Ψ is invariant under lattice translation, i.e., $\Psi([C], \Lambda) = \Psi([C + \mathbf{x}], \Lambda + \mathbf{x})$ for any lattice point \mathbf{x} .
- (P5) Its value on a lattice point is 1, i.e. $\Psi([0], \Lambda) = 1$.
- (P6) Ψ is invariant under orthogonal unimodular transformation; that is, if T is an orthogonal unimodular transformation, for any cone C , we have $\Psi([C], \Lambda) = \Psi([T(C)], T(\Lambda))$.
- (P7) Suppose Λ' is the lattice generated by a subset of a basis of the lattice Λ and C is in the affine span of Λ' . Then $\Psi([C], \Lambda) = \Psi([C], \Lambda')$.
- (P8) It can be computed in polynomial time fixing the dimension.

Remark 3.3. Note that for integral polytopes, we have that $\text{tcone}^p(F, P)$ is a lattice translation of $\text{fcone}^p(F, P)$. Therefore, by Property (P5)

$$(3.4) \quad \alpha(F, P) = \Psi([\text{tcone}^p(F, P)], (V \cap \mathbb{Z}^D)/L) = \Psi([\text{fcone}^p(F, P)], (V' \cap \mathbb{Z}^D)/L'),$$

where V and L are defined as for (3.3), and V' and L' are obtained by shifting V and L to the origin. Because $\text{fcone}^p(F, P)$ is determined by $\text{ncone}(F, P)$ as in (2.3), one sees that $\alpha(F, P)$ depends only on the normal cone of P at F . Therefore, this construction of $\alpha(F, P)$ does give McMullen's formula (3.1).

We have an immediate corollary to Properties (P1) and (P3):

Corollary 3.4. *Suppose C_1, C_2, \dots, C_k and K are cones satisfying*

$$[C] \equiv \sum_{i=1}^k [C_i] \quad \text{modulo polyhedra with lines.}$$

Then

$$\Psi([C], \Lambda) = \sum_{i=1}^k \Psi([C_i], \Lambda).$$

3.2. Reduction theorem. We've already discussed a consequence of the existence of McMullen's formula, which reduce the problem of proving Ehrhart positivity to proving α -positivity. Now we will discuss a very important consequence of the BV- α -valuation – the reduction theorem – using which we complete the proof of Theorem 1.4.

For the rest of the section, we assume $\alpha(F, P)$ comes from the BV- α -valuation. Also, because we only deal with integral polytopes, we will take (3.4) as the definition of $\alpha(F, P)$.

Lemma 3.5. *Suppose V is a subspace of \mathbb{R}^D , and $P \subset V + \mathbf{x}$ and $Q \subset V + \mathbf{y}$ are two integral polytopes for some points \mathbf{x} and \mathbf{y} .*

Let F be a face of P . Suppose there exist faces G_1, G_2, \dots, G_r of Q of the same dimension such that

$$(3.5) \quad \text{ncone}_V(F, P) = \cup_{i=1}^r \text{ncone}_V(G_i, Q).$$

Then F is of the same dimension as G_i 's, and

$$\alpha(F, P) = \sum_{i=1}^r \alpha(G_i, Q).$$

Proof. The first consequence of Equation (3.5) is that $\text{ncone}_V(F, P)$ and $\text{ncone}_V(G_i, Q)$'s all span the same subspace. Let L be the orthogonal complement of this subspace with respect to V . Then by Lemma 2.4, affine spans of F and G_i 's are all shifts of L . Hence, F has the same dimension as G_i 's. Letting $\Lambda := (V \cap \mathbb{Z}^D)/L$, we have

$$\alpha(F, P) = \Psi(\text{fcone}^p(F, P), \Lambda), \quad \text{and} \quad \alpha(G_i, Q) = \Psi(\text{fcone}^p(G_i, Q), \Lambda) \quad \forall i.$$

Next, since $\text{ncone}_V(G_i, Q) \cap \text{ncone}_V(G_j, Q)$ is a lower dimensional cone for any $i \neq j$, we have

$$[\text{ncone}(F, P)] \equiv \sum_{i=1}^r [\text{ncone}(G_i, Q)] \quad \text{modulo polyhedra contained in proper subspaces.}$$

Taking the polar of the above identity and applying (2.3) yields

$$[\text{fcone}^p(F, P)] \equiv \sum_{i=1}^r [\text{fcone}^p(G_i, Q)] \quad \text{modulo polyhedra with lines.}$$

Applying Corollary 3.4, we obtained the desired identity. \square

Theorem 3.6 (Reduction Theorem). *Suppose V is a subspace of \mathbb{R}^D , and $P \subset V + \mathbf{x}$ and $Q \subset V + \mathbf{y}$ are two integral polytopes for some points \mathbf{x} and \mathbf{y} . Assume further the normal fan $\Sigma_V(P)$ of P with respect to V is a refinement of the normal fan $\Sigma_V(Q)$ of Q with respect to V . Then for any fixed k , if $\alpha(F, P) > 0$ for every k -dimensional face F of P , then $\alpha(G, Q) > 0$ for every k -dimensional face G of Q .*

Therefore, BV - α -positivity of P implies BV - α -positivity of Q .

The above reduction theorem and Proposition 2.7 immediately give the following result and complete the proof for Theorem 1.4

Theorem 3.7 (Reduction Theorem, special form). *Let $Q \subset \mathbb{R}^{n+1}$ be a generalized permutohedron. Then for any fixed k , if $\alpha(F, \Pi_n) > 0$ for every k -dimensional face F of Π_n , then $\alpha(G, Q) > 0$ for every k -dimensional face G of Q .*

Therefore, BV - α -positivity of Π_n implies BV - α -positivity of Q .

Proof of Theorem 1.4. The theorem follows from Theorem 3.7 and Lemma 3.2. \square

Remark 3.8. All permutohedra of dimension at most n appear in \mathbb{R}^{n+1} , and Proposition 2.7 applies to all of these permutohedra. Therefore, the polytope Q in Theorem 3.7 could be any permutohedron of dimension up to n .

Remark 3.9. Theorem 3.7 stills holds if we replace Π_n with any generic permutohedron, that is, any $\text{Perm}(\mathbf{v})$ where $\mathbf{v} \in \mathbb{R}^{n+1}$ is a vector with distinct coordinates.

3.3. More applications. Theorem 3.6 mainly follows from Properties (P1) and (P3) of the BV- α -valuation. In this subsection, we will give more applications of the BV- α -valuation as consequences of other properties, in particular Property (P6), showing reasons why this particular construction of α is nice.

Lemma 3.10. *Suppose Λ is a lattice, and P is a nonempty polytope in the affine span of Λ . Then*

$$(3.6) \quad \sum_{\mathbf{v} \in \text{vert}(P)} \Psi(\text{fcone}^P(\mathbf{v}, P), \Lambda) = 1.$$

Let C be a 1-dimensional cone generated by a vector $\mathbf{v} \in \Lambda$. Then

$$(3.7) \quad \Psi(C, \Lambda) = 1/2.$$

Proof. Equation (3.6) follows from Corollary 3.4 and the identity in Theorem 2.8.

Consider the polytope $Q := \text{conv}\{\mathbf{0}, \mathbf{v}\}$, which has two vertices $\mathbf{0}$ and \mathbf{v} and has pointed feasible cones:

$$\text{fcone}^P(\mathbf{0}, Q) = C, \quad \text{fcone}^P(\mathbf{v}, Q) = -C.$$

By (3.6), we have

$$\Psi(C, \Lambda) + \Psi(-C, \Lambda) = 1.$$

However, by Property (P6), we have

$$\Psi(C, \Lambda) = \Psi(-C, -\Lambda) = \Psi(-C, \Lambda).$$

Then (3.7) follows. □

We then apply the above lemma to obtain results on $\alpha(F, P)$.

Lemma 3.11. *Suppose P is a nonempty integral polytope. Then*

$$(3.8) \quad \sum_{\mathbf{v} \in \text{vert}(P)} \alpha(\mathbf{v}, P) = 1;$$

$$(3.9) \quad \alpha(F, P) = 1/2, \quad \text{for any facet } F \text{ of } P;$$

$$(3.10) \quad \alpha(P, P) = 1.$$

Proof. Equalities (3.8) and (3.9) follow from (3.4), Property (P7), and Lemma 3.10. Equality (3.10) follows from the fact that $\text{fcone}^P(P, P) = \mathbf{0}$. □

Remark 3.12. Formulas (3.8) and (3.10) hold for any constructions for $\alpha(F, P)$; however, Formula (3.9) does not hold for all the known constructions for $\alpha(F, P)$, and thus is a special property of the BV- α -valuation.

We use the above lemma to give a quick proof for the following modified version of Pick's theorem, in which one sees that the α -values given above really corresponds to the coefficients appearing in Pick's theorem.

Theorem 3.13 (Pick's theorem). *Let $P \subset \mathbb{Z}^2$ be an integral polygon. Then*

$$\text{Lat}(P) = \text{area}(P) + \frac{1}{2} \text{Lat}(\partial P) + 1,$$

where ∂P denotes the boundary of P .

Proof. Applying Lemma 3.11, we get

$$\text{Lat}(P) = 1 \cdot \text{area}(P) + \frac{1}{2} \cdot \sum_{E: \text{edge of } P} \text{nvols}(E) + 1.$$

It is easy to see that $\sum_{E: \text{edge of } P} \text{nvols}(E)$ is precisely the number of lattice points on the boundary of P . Thus, the theorem follows. \square

We remark that Lemma 3.11 also directly gives the results on the three coefficients of $i(P, t)$ with explicit simple descriptions: the leading coefficient is equal to the normalized volume of P , the second coefficient is one half of the sum of the normalized volumes of facets, and the constant term is always 1. This pattern extend naturally to *boxes*.

Definition 3.14. An D -dimensional *box* is a polytope defined by

$$\{\mathbf{x} \in \mathbb{R}^D : a_i \leq x_i \leq b_i \quad \forall i \in [D]\}$$

for some vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^D$.

Example 3.15. Let P be an D -dimensional box. The pointed feasible cone of P at any vertex is equal to an orthant of \mathbb{R}^D . By Property (P6),

$$\alpha(\mathbf{v}, P) = \Psi(\mathbf{v}, \mathbb{Z}^D) = \Psi(\mathbf{u}, \mathbb{Z}^D) = \alpha(\mathbf{u}, P), \quad \forall \mathbf{v}, \mathbf{u} \in \text{vert}(P).$$

Since P has 2^D vertices and we have Equality (3.8), we conclude

$$\alpha(\mathbf{v}, P) = \frac{1}{2^D} \quad \forall \mathbf{v} \in \text{vert}(P), \quad \text{and} \quad \Psi(C, \mathbb{Z}^D) = \frac{1}{2^D} \quad \text{for any } D\text{-dim orthant } C.$$

Furthermore, note that for any k -dimensional face of P , the pointed feasible cone of P at F is an orthant of \mathbb{R}^{D-k} . So by (3.11),

$$\alpha(F, P) = \Psi(\text{fcone}^P(F, P), \mathbb{Z}^{D-k}) = \frac{1}{2^{D-k}}.$$

Therefore, applying (3.2), we get that the coefficient of t^k in the Ehrhart polynomial $i(P, t)$ of the n -dimensional box P is

$$\frac{1}{2^{D-k}} \sum_{F: \text{a } k\text{-dim face of } P} \text{nvols}(F).$$

Let v_k be the sum of the normalized volumes of all k -dimensional faces. Then

$$i(P, t) = v_D t^D + \frac{1}{2} v_{D-1} t^{D-1} + \frac{1}{4} v_{D-2} t^{D-2} + \cdots + \frac{1}{2^{D-k}} v_k t^k + \cdots + 1.$$

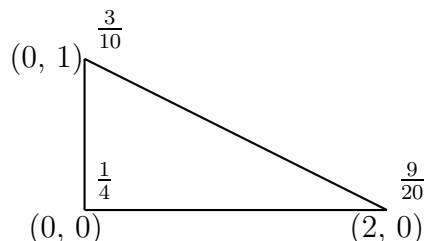
From results we show above, one sees that Property (P6) is an important property for the BV- α -valuation. It provides us a way to obtain α -values for special situations without explicit computation for Ψ which could be quite complicated. (See examples in the next subsection.) In fact, the following lemma, which states a special case of this property, will be applied extensively when we compute α -values for regular permutohedra in Section 4.

Lemma 3.16. *The valuation Ψ is symmetric about the coordinates, i.e., for any cone $C \in \mathbb{R}^D$ and any permutation $\sigma \in \mathfrak{S}_D$, we have*

$$\Psi(C, \Lambda) = \Psi(\sigma(C), \sigma(\Lambda)),$$

where $\sigma(T) = \{(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(D)}) : (x_1, \dots, x_D) \in T\}$ for any set $T \subseteq \mathbb{R}^D$.

FIGURE 1. The vertices add 1 in a nontrivial way



Proof. Let M_σ be the permutation matrix corresponding to σ . Then the lemma follows from the observation that T is mapped to $\sigma(T)$ under the linear transformation M_σ and any permutation matrix is orthonormal and unimodular. \square

The above result motivates the following definition.

Definition 3.17. Suppose α is a construction such that McMullen's formula (3.1) holds. We say it is *symmetric about the coordinates*

$$\alpha(F, P) = \alpha(G, Q)$$

whenever $\text{fcone}^P(F, P) = \sigma(\text{fcone}^P(G, Q))$ for some $\sigma \in \mathfrak{S}_n$.

Therefore, we have the following:

Lemma 3.18. *The BV- α -valuation is symmetric about the coordinates.*

Property (P6), in particular Lemma 3.18, does not hold for all the known Ψ or α -constructions. For example, the construction given by Pommersheim and Thomas in [12] depends on an ordering of a basis for the vector space, which means their construction is not symmetric about coordinates. This is one of the reasons why we work with the BV- α -valuations for this paper.

3.4. Examples of computing Ψ . Below we will give examples of computing Ψ and using which to get $\alpha(F, P)$. The computation of the function Ψ associated to Berline-Vergne's construction is carried out recursively. Hence, it is quicker to compute Ψ for lower dimensional cones. Since the dimension of $\text{fcone}^P(F, P)$ is equal to the codimension of F with respect to P , the value of $\alpha(F, P)$ is easier to compute if F is a higher dimensional face.

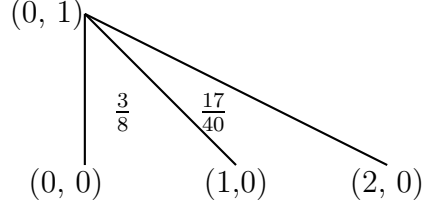
In general, the computation of $\Psi(C, \Lambda)$ is quite complicated. However, when C is a unimodular cone (with respect to Λ), that is, C is generated by a basis of Λ , computations are greatly simplified. In small dimensions we can even give a simple closed expression for Ψ of unimodular cones. The following result is given in [2, Example 19.3].

Lemma 3.19. *If $C = \text{Cone}(\mathbf{u}_1, \mathbf{u}_2)$, where $\{\mathbf{u}_1, \mathbf{u}_2\}$ is a basis for a lattice Λ , then*

$$\Psi(C, \Lambda) = \frac{1}{4} + \frac{1}{12} \left(\frac{\langle \mathbf{u}_1, \mathbf{u}_2 \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle} + \frac{\langle \mathbf{u}_1, \mathbf{u}_2 \rangle}{\langle \mathbf{u}_2, \mathbf{u}_2 \rangle} \right).$$

Example 3.20. Consider the polygon P in \mathbb{R}^2 with vertices $\mathbf{v}_1 = (0,0)$, $\mathbf{v}_2 = (2,0)$, and $\mathbf{v}_3 = (0,1)$. The pointed feasible cone of P at \mathbf{v}_1 is the first quadrant, with Ψ value

FIGURE 2. Unimodular decomposition of C_3



1/4 (see Example 3.15). The pointed feasible cone of P at \mathbf{v}_2 is the unimodular cone $C_2 = \text{Cone}((-2, 1), (-1, 0))$. Applying Lemma 3.19,

$$\alpha(\mathbf{v}_2, P) = \Psi(C_2, \mathbb{Z}^2) = \frac{1}{4} + \frac{1}{12} \left(\frac{2}{5} + \frac{2}{1} \right) = \frac{9}{20}.$$

The pointed feasible cone of P at \mathbf{v}_3 is the cone $C_3 = \text{Cone}((0, -1), (2, -1))$, which is not unimodular, so we cannot directly apply Lemma 3.19 to compute $\Psi(C_3, \mathbb{Z}^2)$. In order to compute it, we first decompose C_3 in the algebra of cones $\mathcal{C}_0(\mathbb{R}^2)$:

$$[C_3] = [\text{Cone}((0, -1), (1, -1))] + [\text{Cone}((1, -1), (2, -1))] - [\text{Cone}((1, -1))]]$$

We apply Lemma 3.19 to the two first cones in the above decomposition and get Ψ values of 3/8 and 17/40. Then applying Lemma 3.10, we get Ψ value of the last cone is 1/2. Finally, by Property (P1), we get

$$\alpha(\mathbf{v}_3, P) = \Psi(C_3, \mathbb{Z}^2) = \frac{3}{8} + \frac{17}{40} - \frac{1}{2} = \frac{3}{10}.$$

We can also verify Equality (3.8):

$$\sum_{\mathbf{v} \in \text{vert}(P)} \alpha(\mathbf{v}, P) = \frac{1}{4} + \frac{9}{20} + \frac{3}{10} = 1$$

We finish this part with a formula for computing Ψ of a 3-dim unimodular cone, which was computed from Maple code.

Lemma 3.21. *If $C = \text{Cone}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ where $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ is a basis for a lattice Λ , then*

$$\Psi(C, \Lambda) = \frac{1}{8} + \frac{1}{24} \left(\frac{\langle \mathbf{u}_1, \mathbf{u}_2 \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle} + \frac{\langle \mathbf{u}_1, \mathbf{u}_2 \rangle}{\langle \mathbf{u}_2, \mathbf{u}_2 \rangle} + \frac{\langle \mathbf{u}_1, \mathbf{u}_3 \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle} + \frac{\langle \mathbf{u}_1, \mathbf{u}_3 \rangle}{\langle \mathbf{u}_3, \mathbf{u}_3 \rangle} + \frac{\langle \mathbf{u}_3, \mathbf{u}_2 \rangle}{\langle \mathbf{u}_2, \mathbf{u}_2 \rangle} + \frac{\langle \mathbf{u}_3, \mathbf{u}_2 \rangle}{\langle \mathbf{u}_3, \mathbf{u}_3 \rangle} \right).$$

Remark 3.22. The formulas for 2-dim and 3-dim unimodular cones appear to be simple. However, the apparent simplicity breaks down for dimension 4. The formula for 4-dim unimodular cones include (way) more than 1000 terms.

4. GENERIC PERMUTOHEDRON

Since the proof of Theorem 1.4 is completed in the last section, we only need to focus on the BV- α -valuation arising from the regular permutohedron Π_n , or any generic permutohedron. (See Remark 3.9.)

We use the following setup.

Setup 4.1. (i) Let $\mathbf{v} = (v_1, v_2, \dots, v_{n+1})$ be a vector with strictly increasing entries, and consider the generalized permutohedron

$$\text{Perm}(\mathbf{v}) = \text{Perm}(v_1, v_2, \dots, v_{n+1}).$$

(ii) Suppose α is a construction such that McMullen's formula (3.1) holds and it is symmetric about the coordinates (see Definition 3.17).

It is clear that (i) covers all generic permutohedron, and the BV- α -valuation is a special case of (ii). Under this setup, we will analyze formula for computing $\text{Lat}(\text{Perm}(\mathbf{v}))$ further, and derive a more combinatorial formula for computing α -values arising from $\text{Perm}(\mathbf{v})$.

Applying McMullen's formula to $P = \text{Perm}(\mathbf{v})$, we get

$$(4.1) \quad \text{Lat}(\text{Perm}(\mathbf{v})) = \sum_{F: \text{ a face of } \text{Perm}(\mathbf{v})} \alpha(F, \text{Perm}(\mathbf{v})) \text{ nvol}(F).$$

Because of the symmetric properties of $\text{Perm}(\mathbf{v})$ and α , there are a lot of terms in the above summand coincident, and it is natural to group them together. In order to this, we need the following definition and proposition.

Definition 4.2. The symmetric group \mathfrak{S}_{n+1} acts linearly on \mathbb{R}^{n+1} by permuting the coordinates. Two subsets $A_1, A_2 \subset \mathbb{R}^{n+1}$ are said to be *symmetric* if they lie in the same orbit, i.e. if there exist $\sigma \in \mathfrak{S}_{n+1}$ such that $\sigma(A_1) = A_2$. Since the action is orthogonal, two symmetric sets are congruent, in particular, they have the same volume (if measurable).

The following results are given in [?].

Proposition 4.3. *There is a one-to-one correspondence between ordered set partitions of $[n+1]$ and faces of $\text{Perm}(\mathbf{v})$ defined as follows:*

For any ordered set partition $\mathcal{P} = (P_1, P_2, \dots, P_l)$ of $[n+1]$, the corresponding face is obtained by maximizing any linear functional given by a vector $\mathbf{c} \in \mathbb{R}^{n+1}$ with the property that

- a) $c_i = c_j$ if i and j are both in P_k for some k , and
- b) $c_i < c_j$ if $i \in P_{k_1}$ and $j \in P_{k_2}$ with $k_1 < k_2$.

Let $m_i = |P_i|$. Then the corresponding face has dimension $n+1-l$ and it is congruent to

$$\text{Perm}(\mathbf{v}_{M_1}) \times \text{Perm}(\mathbf{v}_{M_2}) \times \dots \times \text{Perm}(\mathbf{v}_{M_l}),$$

where $\mathbf{v}_{M_i} = \left(v_j : \sum_{k=1}^{i-1} m_k < j \leq \sum_{k=1}^i m_k \right)$. In other words, \mathbf{v}_{M_1} consists of the first m_1 entries of $\mathbf{v} = (v_1, \dots, v_{n+1})$, \mathbf{v}_{M_2} consists of the next m_2 entries, and so on.

We call the ordered tuple $\mathbf{m} := (m_1, m_2, \dots, m_l)$ the composition of \mathcal{P} .

Example 4.4. Let $n = 5$ and consider the ordered set partition $\mathcal{P} = (\{1, 4, 6\}, \{2, 5\}, \{3\})$. Then the composition of \mathcal{P} is $(3, 2, 1)$.

The face of $\text{Perm}(\mathbf{v})$ corresponding to \mathcal{P} is the face which optimizes any linear functional $\mathbf{c} = (c_1, c_2, c_3, c_4, c_5, c_6)$ with $c_1 = c_4 = c_6 < c_2 = c_5 < c_3$. In order to figure out this corresponding face, we look for vertices of $\text{Perm}(\mathbf{v})$ optimizing such a functional \mathbf{c} . One sees that v_1, v_2 and v_3 should be in positions 1, 4 and 6 of these vertices, v_4 and v_5 in positions 2 and 5, and v_6 in position 3. Therefore, the desired vertices are

$$\left\{ (v_{\mu(1)}, v_{\tau(1)}, v_6, v_{\mu(2)}, v_{\tau(2)}, v_{\mu(3)}) : \mu \in \mathfrak{S}_{\{1,2,3\}}, \tau \in \mathfrak{S}_{\{4,5\}} \right\}.$$

Hence, we conclude that the face that is corresponding to the ordered set partition $\mathcal{P} = (\{1, 4, 6\}, \{2, 5\}, \{3\})$ is congruent to

$$\text{Perm}(v_1, v_2, v_3) \times \text{Perm}(v_4, v_5) \times \text{Perm}(v_6).$$

By Proposition 4.3, two faces of $\text{Perm}(\mathbf{v})$ are in the same orbit, i.e. they are symmetric, if and only if their corresponding ordered set partitions have the same composition. Therefore, the orbits of the \mathfrak{S}_{n+1} -action on the faces of $\text{Perm}(\mathbf{v})$ are indexed by compositions \mathbf{m} of $n+1$. We denote the orbit corresponding to the composition \mathbf{m} by $\mathcal{O}_n(\mathbf{m})$.

Furthermore, under Setup 4.1, the construction α is symmetric about the coordinates. Hence, for any fixed \mathbf{m} , the value $\alpha(F, \text{Perm}(\mathbf{v}))$ is a constant on $\mathcal{O}_n(\mathbf{m})$, and thus we can define $\alpha_n(\mathbf{m})$ to be this constant.

Finally, a canonical representative of $\mathcal{O}_n(\mathbf{m})$ is chosen as below.

Definition 4.5. Let $\mathbf{m} = (m_1, m_2, \dots, m_l)$ be a composition of $n+1$. Define an ordered set partition $\mathcal{P}(\mathbf{m}) = (\mathcal{P}(\mathbf{m})_i)$ where

$$\mathcal{P}(\mathbf{m})_i = \left[\sum_{k=1}^{i-1} m_k + 1, \sum_{k=1}^i m_k \right].$$

In other words, the first subset $\mathcal{P}(\mathbf{m})_1$ consists of the first m_1 positive integers, the second subset $\mathcal{P}(\mathbf{m})_2$ consists of the next m_2 positive integers, and so on.

Then we define $F_{\mathbf{m}}$ to be the face corresponding to the ordered set partition $\mathcal{P}(\mathbf{m})$ under the bijection given in Proposition 4.3.

Example 4.6. Let $n = 5$ and $\mathbf{m} = (3, 2, 1)$. Then $\mathcal{P}(\mathbf{m}) = (\{1, 2, 3\}, \{4, 5\}, \{6\})$, and

$$(4.2) \quad F_{\mathbf{m}} = \text{conv} \left\{ (v_{\mu(1)}, v_{\mu(2)}, v_{\mu(3)}, v_{\tau(1)}, v_{\tau(2)}, v_6) : \mu \in \mathfrak{S}_{\{1,2,3\}}, \tau \in \mathfrak{S}_{\{4,5\}} \right\}.$$

Applying the above discussions to (4.1), we get

$$(4.3) \quad \text{Lat}(\text{Perm}(\mathbf{v})) = \sum_{\mathbf{m} : \text{composition of } n+1} |\mathcal{O}_n(\mathbf{m})| \alpha_n(\mathbf{m}) \text{nvol}(F_{\mathbf{m}}).$$

Note that one of the terms in the above formula can be explicitly described: For a fixed $\mathbf{m} = (m_1, \dots, m_l)$, the number of faces in $\mathcal{O}_n(\mathbf{m})$ is equal to the number of ordered set partitions whose compositions are \mathbf{m} . Thus,

$$(4.4) \quad |\mathcal{O}_n(\mathbf{m})| = \binom{n+1}{m_1, m_2, \dots, m_l}.$$

Next, we investigate properties of the face $F_{\mathbf{m}}$. It is easy to see that $F_{\mathbf{m}}$ is always adjacent to the vertex $\mathbf{v} = (v_1, \dots, v_n, v_{n+1})$. Note that the (pointed) feasible cone of $\text{Perm}(\mathbf{v})$ at \mathbf{v} is spanned by the following n vectors:

$$\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_2 - \mathbf{e}_3, \dots, \mathbf{e}_n - \mathbf{e}_{n+1}.$$

Hence, subsets of these n -vectors are in one-to-one correspondence to faces of $\text{Perm}(\mathbf{v})$ that are adjacent to \mathbf{v} . This motivates the following definition.

Definition 4.7. Let \mathbf{m} be a composition of $n+1$, and $F_{\mathbf{m}}$ the face of $\text{Perm}(\mathbf{v})$ that is associated to \mathbf{m} . Define $S = \mathcal{S}(\mathbf{m})$ to be the subset of $[n]$ such that

$$\text{affine span of } F_{\mathbf{m}} = \mathbf{v} + \text{span}\{\mathbf{e}_i - \mathbf{e}_{i+1} : i \in S\}.$$

Example 4.8. Let $n = 5$ and $\mathbf{m} = (3, 2, 1)$. Then $F_{\mathbf{m}}$ is given by (4.2). One checks that

$$\text{affine span of } F_{\mathbf{m}} = \mathbf{v} + \text{span}\{\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_2 - \mathbf{e}_3, \mathbf{e}_4 - \mathbf{e}_5\}.$$

Hence, $S = \mathcal{S}(\mathbf{m}) = \{1, 2, 4\}$.

Note that both compositions of $n + 1$ and subsets of $[n]$ are counted by the same number, 2^n . So it is natural to ask whether the map $\mathbf{m} \mapsto \mathcal{S}(\mathbf{m})$ is a bijection between these two families of objects. Indeed, we have the following lemma.

Lemma 4.9. *The map \mathcal{S} defined in Definition 4.7 is a bijection from compositions of $n + 1$ to subsets of $[n]$.*

Proof. we can define an inverse to \mathcal{S} in the following way: Suppose S is a subset of $[n]$. Let $T = [n + 1] \setminus S$. Suppose $T = \{t_1 < t_2 < \dots < t_l\}$. Then one verifies that

$$S \mapsto \mathbf{m} := (t_1, t_2 - t_1, \dots, t_l - t_{l-1})$$

is an inverse to \mathcal{S} , completing the proof. \square

The above lemma tells us that the term $F_{\mathbf{m}}$ appearing in the summand of Formula (4.3) can also be understood as faces of $\text{Perm}(\mathbf{v})$ that are adjacent to the vertex \mathbf{m} . Furthermore, abusing notation we can use subsets of $[n]$ to index Formula (4.3):

$$(4.5) \quad \text{Lat}(\text{Perm}(\mathbf{v})) = \sum_{S \subseteq [n]} |\mathcal{O}_n(S)| \alpha_n(S) \text{nvol}(F_S).$$

Note that $\dim(F_S) = |S|$. In fact, for arguments we will carry out in both Section 5 and Section 6, it is more convenient to use subsets of $[n]$ as our indexing.

5. EVIDENCE TO CONJECTURES

We will explore consequences of Formula (4.5) further in Section 6, and will only focus on computing $\alpha_n(S)$ for some special cases in this section. The main results of this section are the following two theorems, providing evidence to our conjectures.

Theorem 5.1. *For all $n \leq 6$, the regular permutohedron Π_n is BV- α -positive.*

Therefore, all the integral generalized permutohedra (including matroid base polytopes) of dimension at most 6 are Ehrhart positive.

Theorem 5.2. *For any n , and any face F of Π_n of codimension 2 or 3, we have $\alpha(F, \Pi_n)$ is positive, where α is the BV- α -valuation.*

Hence, the third and fourth coefficients of the Ehrhart polynomial of any integral generalized permutohedra (including matroid base polytopes) are positive.

Clearly, in order to prove the above two theorems, we just need to verify the following two statements respectively, assuming α is the BV- α -valuation:

$$(5.1) \quad \alpha_n(S) > 0, \quad \forall S \subseteq [n], \forall n \leq 6,$$

$$(5.2) \quad \alpha_n(S) > 0, \quad \forall S \subset [n], |S| = n - 2, n - 3.$$

5.1. How to Compute $\alpha_n(S)$. We first describe how to compute $\alpha_n(S)$. Let V be the n -dimensional subspace of \mathbb{R}^{n+1} determined by $x_1 + x_2 + \cdots + x_{n+1} = 0$. For any $S \subseteq [n]$, let $L_S := \text{span}\{\mathbf{e}_i - \mathbf{e}_{i+1} : i \in S\}$. By Definition 4.7,

$$\text{affine span of } F_S = \mathbf{v} + L_S.$$

Therefore, applying Formula (3.4) to our situation, we get

$$(5.3) \quad \alpha_n(S) = \Psi(\text{fcone}^p(F_S, \Pi_n), \Lambda_S),$$

where

$$\Lambda_S := (V \cap \mathbb{Z}^{n+1})/L_S.$$

Observe that $\{\mathbf{e}_i - \mathbf{e}_{i+1} : i \in [n]\}$ is a unimodular basis for V . Hence, Λ_S is the lattice spanned by projections of vectors in

$$\{\mathbf{e}_i - \mathbf{e}_{i+1} : i \in [n] \setminus S\}$$

to L_S^\perp , the orthogonal complement of L_S in V . Furthermore, $\text{fcone}^p(F_S, \Pi_n)$ is the span of these projection vectors. We then apply directly the Berline Vergne's recursive definition of Ψ to (5.3) to find $\alpha_n(S)$.

5.2. Small dimensions. We now prove (5.1) by computing $\alpha_n(S)$ directly, which implies Theorem 5.1.

When $n = 1, 2$, $\alpha_n(S)$ corresponds to $\alpha(F, \Pi_n)$ where F is either Π_n or a facet of Π_n . Thus, the positivity of $\alpha_n(S)$ follows from Lemma 3.11. Hence, we only need to compute $\alpha_n(S)$ for $3 \leq n \leq 6$. We use the procedure described above to calculate the values of these $\alpha_n(S)$ and summarize in the examples below.

Example 5.3. For $n = 3$:

S	\emptyset	1	2	3	12	13	23	123
$\alpha_3(S)$	1/24	11/72	7/36	11/72	1/2	1/2	1/2	1

Example 5.4. For $n = 4$:

S	\emptyset	1	2	3	4	12	13	14
$\alpha_4(S)$	1/120	5/144	7/144	7/144	5/144	7/48	13/72	5/36
S	23	24	34	123	124	134	234	1234
$\alpha_4(S)$	5/24	13/72	7/48	1/2	1/2	1/2	1/2	1

Example 5.5. For $n = 5$:

S	\emptyset	1			2		3		4		5	
$\alpha_5(S)$	1/720	137/21600			101/10800		37/3600		101/10800		137/21600	
S	12	13	14	15	23	24	25	34	35	45		
$\alpha_5(S)$	1/32	1/24	1/24	1/36	5/96	1/18	1/24	5/96	1/24	1/32		
S	123	124	125		134		135	145	234			
$\alpha_5(S)$	17/120	31/180	19/144		47/240		1/6	19/144	13/60			
S	235	245	345		1234	1235	1245	1345	2345	12345		
$\alpha_5(S)$	47/240	31/180	17/120		1/2	1/2	1/2	1/2	1/2	1		

Example 5.6. For $n = 6$:

S	\emptyset	1	2	3	4	5	6
$\alpha_6(S)$	1/5040	7/7200	1/675	37/21600	37/21600	1/675	7/7200

S	12	13	14	15	16	23			
$\alpha_6(S)$	29/5400	1/135	541/64800	149/21600	151/32400	211/21600			
S	24	25	26	34	35	36			
$\alpha_6(S)$	719/64800	181/16200	149/21600	41/3600	719/64800	541/64800			
S	45	46	56	123	124	125	126	134	135
$\alpha_6(S)$	211/21600	1/135	29/5400	7/240	3/80	11/288	7/288	11/240	7/144
S	136	145	146	156	234	235	236		
$\alpha_6(S)$	5/144	13/288	5/144	7/288	13/240	17/288	13/288		
S	245	246	256	345	346	356	456		
$\alpha_6(S)$	17/288	7/144	11/288	13/240	11/240	3/80	7/240		
S	1234	1235	1236	1245	1246	1256	1345	1346	
$\alpha_6(S)$	5/36	1/6	23/180	3/16	19/120	1/8	37/180	11/60	
S	1356	1456	2345	2346	2356	2456	3456		
$\alpha_6(S)$	19/120	23/180	2/9	37/180	3/16	1/6	5/36		
S	12345	12346	12356	12456	13456	23456	123456		
$\alpha_6(S)$	1/2	1/2	1/2	1/2	1/2	1/2	1		

One sees in all the examples above that $\alpha_n(S)$ is always positive for $n \leq 6$. Hence, we complete our proof for Theorem 5.1.

Remark 5.7. By Remark 3.8, if $\alpha_6(S)$ is positive for all S , then we immediately have $\alpha_n(S) > 0$ for any $n < 6$ and any S . Hence, it is enough to use Example 5.6 to prove Theorem 5.1. We include all the other examples so that we have more data for $\alpha_n(S)$, which might be helpful for future work.

5.3. Top coefficients. We then prove (5.2), which implies Theorem 5.2.

We will again apply the procedure described in Subsection 5.1 to compute $\alpha_n(S)$.

We start by considering the cases where $|S| = n - 2$. Suppose $[n] \setminus S = \{i, j\}$ with $i < j$. Then $L_S = \text{span}(\mathbf{e}_k - \mathbf{e}_{k+1} : k \neq i, j)$. For $k = i, j$, let \mathbf{u}_k be the projection of $\mathbf{e}_k - \mathbf{e}_{k+1}$ to L_S^\perp , the orthogonal complement of L_S . Then

$$\mathbf{u}_i = \left(\underbrace{\frac{1}{i}, \frac{1}{i}, \dots, \frac{1}{i}}_i, \underbrace{\frac{-1}{j-i}, \frac{-1}{j-i}, \dots, \frac{-1}{j-i}}_{j-i}, \underbrace{0, 0, \dots, 0}_{n+1-j} \right),$$

$$\mathbf{u}_j = \left(\underbrace{0, 0, \dots, 0}_i, \underbrace{\frac{1}{j-i}, \frac{1}{j-i}, \dots, \frac{1}{j-i}}_{j-i}, \underbrace{\frac{-1}{n+1-j}, \frac{-1}{n+1-j}, \dots, \frac{-1}{n+1-j}}_{n+1-j} \right).$$

Note that $\{\mathbf{u}_i, \mathbf{u}_j\}$ is a basis of Λ_S and also spans $\text{fcone}^P(F_S, \Pi_n)$. Therefore, applying Lemma 3.19, we obtain a precise formula for $\alpha_n(S)$.

Lemma 5.8. *Suppose $S \subset [n]$ such that $[n] \setminus S = \{i, j\}$ with $i < j$. Then*

$$(5.4) \quad \alpha_n(S) = \frac{1}{4} - \frac{1}{12} \left(\frac{i}{j} + \frac{n+1-j}{n+1-i} \right)$$

We repeat the same procedure for the cases where $|S| = n - 3$. Suppose $[n] \setminus S = \{i, j, k\}$ with $i < j < k$. Then $L_S = \text{span}(\mathbf{e}_l - \mathbf{e}_{l+1} : l \neq i, j)$. For $l = i, j, k$, let \mathbf{u}_l be the projection of $\mathbf{e}_l - \mathbf{e}_{l+1}$ to L_S^\perp , the orthogonal complement of L_S . Then

$$\begin{aligned} \mathbf{u}_i &= \left(\underbrace{\frac{1}{i}, \frac{1}{i}, \dots, \frac{1}{i}}_i, \underbrace{\frac{-1}{j-i}, \frac{-1}{j-i}, \dots, \frac{-1}{j-i}}_{j-i}, \underbrace{0, 0, \dots, 0}_{n+1-j} \right), \\ \mathbf{u}_j &= \left(\underbrace{0, \dots, 0}_i, \underbrace{\frac{1}{j-i}, \dots, \frac{1}{j-i}}_{j-i}, \underbrace{\frac{-1}{k-j}, \dots, \frac{-1}{k-j}}_{k-j}, \underbrace{0, \dots, 0}_{n+1-k} \right), \\ \mathbf{u}_k &= \left(\underbrace{0, \dots, 0}_j, \underbrace{\frac{1}{k-j}, \dots, \frac{1}{k-j}}_{k-j}, \underbrace{\frac{-1}{n+1-k}, \dots, \frac{-1}{n+1-k}}_{n+1-k} \right). \end{aligned}$$

Here $\{\mathbf{u}_i, \mathbf{u}_j, \mathbf{u}_k\}$ is a basis of Λ_S and also spans $\text{fcone}^p(F_S, \Pi_n)$. Therefore, applying Lemma 3.21, we obtain a precise formula for $\alpha_n(S)$.

Lemma 5.9. *Suppose $S \subset [n]$ such that $[n] \setminus S = \{i, j, k\}$ with $i < j < k$. Then*

$$(5.5) \quad \alpha_n(S) = \frac{1}{8} - \frac{1}{24} \left(\frac{i}{j} + 1 + \frac{n+1-k}{n+1-j} \right)$$

It is easy to check that $\alpha_n(S)$ are positive in both (5.4) and (5.5). Hence, Theorem 5.2 follows.

Remark 5.10. Similar as to the proof of Theorem 5.1, it is not necessary to prove $\alpha_n(S)$ for both $|S| = n - 2$ and $|S| = n - 3$. In fact, it follows from Remark 3.8 that if $\alpha_n(S) > 0$ for all n and S of size $n - 3$, then $\alpha_n(S) > 0$ for all n and S of size $n - 2$. We again include the cases of $|S| = n - 2$ to provide more data of $\alpha_n(S)$.

6. UNIQUENESS

In this section, we take a different point of view and investigate the uniqueness of the Ψ/α constructions for McMullen's formula. We will apply the mixed valuation theories introduced in Subsection 2.4 to Minkowski sums of *hypersimplices*.

Definition 6.1. The *hypersimplex* $\Delta_{k,n+1}$ is defined as

$$\Delta_{k,n+1} = \text{Perm}(\underbrace{0, \dots, 0}_{n+1-k}, \underbrace{1, \dots, 1}_k).$$

The main result of this section is that the α -values of faces of $\text{Perm}(\mathbf{v})$ are uniquely determined as a scalar of mixed valuation of hypersimplices if we require α and \mathbf{v} to be given under Setup 4.1 (Theorem 6.5). Furthermore, as a consequence of this result, we give an equivalent statement of Conjecture 1.3 in Corollary 6.6.

As in Setup 4.1, we consider the generalized permutohedron $\text{Perm}(\mathbf{v}) = \text{Perm}(v_1, v_2, \dots, v_n, v_{n+1})$ with $v_1 < v_2 < \dots < v_n < v_{n+1}$. We have the following expression as Minkowski sum [14, Section 16].

$$\text{Perm}(\mathbf{v}) = w_1 \Delta_{1,n+1} + w_2 \Delta_{2,n+1} + \dots + w_n \Delta_{n,n+1},$$

where

$$(6.1) \quad w_i := v_{i+1} - v_i \text{ for } i = 1, 2, \dots, n.$$

(The w_i 's are actually edge lengths of edges of $\text{Perm}(\mathbf{v})$. But this is not relevant to our discussion.)

Using the results on mixed volumes – Theorems 2.12 and 2.13 – we have the following:

Lemma 6.2. *The normalized volume of $\text{Perm}(\mathbf{v})$ is a homogeneous polynomial in w_i 's with strictly positive coefficients.*

In [14], the coefficients of the above homogeneous polynomial are called *mixed Eulerian numbers*, and some basic properties are established. One of the properties is the following:

Lemma 6.3. *The coefficient of $w_1 w_2 \dots w_n$, the unique squarefree monomial, in the homogeneous polynomial assumed in Lemma 6.2 is $n!$*

Note that Lemma 6.3 is equivalent to

$$(6.2) \quad \sum_{\sigma \in \mathfrak{S}_n} \mathcal{MLat}^n(\Delta_{\sigma(1),n+1}, \Delta_{\sigma(2),n+1}, \dots, \Delta_{\sigma(n),n+1}) = n! \\ \iff \mathcal{MLat}^n(\Delta_{1,n+1}, \Delta_{2,n+1}, \dots, \Delta_{n,n+1}) = 1,$$

where the “ \iff ” follows from (2.6).

Recall in Section 4, we associate a face $F_{\mathbf{m}}$ of $\text{Perm}(\mathbf{v})$ to any composition \mathbf{m} of $n+1$, establish a bijection \mathcal{S} from \mathbf{m} to subsets S of $[n]$, and rewrite $F_{\mathbf{m}}$ as F_S . We have the following result on the normalized volume of F_S .

Proposition 6.4. *Suppose $P = \text{Perm}(\mathbf{v})$ and $S \subseteq [n]$. Let F_S be the corresponding face of P as defined in Section 4, and $\mathbf{m} = (m_1, \dots, m_l) := \mathcal{S}^{-1}(S)$ is the composition in bijection to S . Then $\text{nvol}(F_S)$ is a homogenous polynomial in $\{w_i : i \in S\}$, whose coefficient of $\prod_{i \in S} w_i$ – the unique squarefree monomial – is*

$$(6.3) \quad C_n(S) := (m_1 - 1)!(m_2 - 1)! \dots (m_l - 1)!,$$

Proof. By Proposition 4.3, the face F_S is congruent to

$$\text{Perm}(\mathbf{v}_{M_1}) \times \text{Perm}(\mathbf{v}_{M_2}) \times \dots \times \text{Perm}(\mathbf{v}_{M_l}),$$

where $\mathbf{v}_{M_i} = \left(v_j : \sum_{k=1}^{i-1} m_k < j \leq \sum_{k=1}^i m_k \right)$. Hence, $\text{nvol}(F_S) = \prod_{i=1}^l \text{nvol}(\text{Perm}(\mathbf{v}_{M_i}))$. Let

$$T_i := \left\{ j : \sum_{k=1}^{i-1} m_k < j < \sum_{k=1}^i m_k \right\}.$$

Then by Lemmas 6.2 and 6.3, the normalized volume of $\text{Perm}(\mathbf{v}_{M_i})$ is a homogeneous polynomial in $\{w_j : j \in T_i\}$, and the coefficient of $\prod_{j \in T_i} w_j$ – the unique squarefree monomial – in this homogeneous polynomial is $(m_i - 1)!$. Therefore, the conclusion follows if we can show

that $S = \cup_{i=1}^l T_i$. However, by the proof of Lemma 4.9, the complement of S with respect to $[n+1]$ is

$$[n+1] \setminus S = \left\{ \sum_{k=1}^i m_i : i = 1, 2, \dots, l \right\},$$

which is exactly $[n+1] \setminus \cup_{i=1}^l T_i$. Hence, we are done. \square

The following is the main result of this section.

Theorem 6.5. *Suppose α and \mathbf{v} are as in Setup 4.1. Then the α values of faces of $\text{Perm}(\mathbf{v})$ are uniquely determined. More specifically, if $S = \{s_1, s_2, \dots, s_k\}$, we have*

$$(6.4) \quad \alpha_n(S) = \frac{1}{C_n(S)|\mathcal{O}_n(S)|} k! \mathcal{MLat}^k(\Delta_{s_1, n+1}, \Delta_{s_2, n+1}, \dots, \Delta_{s_k, n+1}),$$

where $C_n(S)$ is defined in (6.3).

In particular the above formula applies to the BV- α -valuation.

One sees that the above theorem gives a connection between the α arising from the regular permutohedron and the mixed lattice points counter function \mathcal{MLat}^k on hypersimplices. Therefore, we have the following:

Corollary 6.6. *The following are equivalent:*

(1) *For any $S = \{s_1, \dots, s_k\} \subseteq [n]$, we have*

$$\mathcal{MLat}^k(\Delta_{s_1, n+1}, \Delta_{s_2, n+1}, \dots, \Delta_{s_k, n+1}).$$

(2) *The regular permutohedron Π_n is BV- α -positive.*

Proof of Theorem 6.5. Let w_i be defined as in (6.1). Theorem 2.15 or Theorems 2.9 and 2.14 tell us that the number of lattice points in

$$\text{Perm}(\mathbf{v}) = w_1 \Delta_{1, n+1} + w_2 \Delta_{2, n+1} + \dots + w_n \Delta_{n, n+1}$$

is a polynomial in the w_i variables. We denote this polynomial by $E = E(w_1, w_2, \dots, w_n)$. We focus on the coefficient of squarefree monomials $w_S := \prod_{i \in S} w_i$ in E . On the one hand, by (2.5) and (2.6), this coefficient is equal to

$$(6.5) \quad k! \mathcal{MLat}^k(\Delta_{s_1, n+1}, \Delta_{s_2, n+1}, \dots, \Delta_{s_k, n+1}).$$

Next by Equation (4.5), we have

$$E(w_1, \dots, w_n) = \sum_{S \subseteq [n]} |\mathcal{O}_n(S)| \alpha_n(S) \text{nvol}(F_S).$$

(Proposition 6.4 guarantees that the right hand side of the above expression is indeed polynomial on the w_i variables.) Note that according to Proposition 6.4, the only contribution to the monomial $w_S = \prod_{i \in S} w_i$ in the summand above is the term corresponding to S , and it is given by $C_n(S)$. Using these, we conclude that the coefficient of w_S in $E(w_1, \dots, w_n)$ is

$$(6.6) \quad \alpha_n(S) |\mathcal{O}_n(S)| C_n(S)$$

Finally, our two expressions, (6.5) and (6.6), for the coefficient of w_S in E has to agree. Hence, the conclusion follows. \square

Proposition 6.4 and Equation (4.4) allow us to give an explicit formula for $|\mathcal{O}_n(S)|C_n(S)$ using the bijection of Definition 4.7.

$$(6.7) \quad \frac{1}{|\mathcal{O}_n(S)|C_n(S)} = \frac{m_1 \cdot m_2 \cdots m_l}{(n+1)!}.$$

Hence, we rewrite Equation (6.4) as

$$(6.8) \quad \alpha_n(S) = \frac{m_1 \cdot m_2 \cdots m_l}{(n+1)!} k! \mathcal{MLat}^k(\Delta_{s_1, n+1}, \Delta_{s_2, n+1}, \dots, \Delta_{s_k, n+1}).$$

Formula (6.8) allows us to compute some examples of $\alpha_n(S)$.

Example 6.7. Consider $S = [n]$ as a subset of $[n+1]$. Its corresponding composition is $\mathbf{m} = \mathcal{S}^{-1}(S) = (n+1)$, and $F_S = \text{Perm}(\mathbf{v})$. By Equation (3.10), we have $\alpha_n(S) = 1$. Therefore, Formula (6.8) gives:

$$\frac{n+1}{(n+1)!} n! \mathcal{MLat}^n(\Delta_{1, n+1}, \Delta_{2, n+1}, \dots, \Delta_{n, n+1}) = 1,$$

agreeing with (6.2).

Example 6.8. Consider $n = 3$ and $S = \{1, 3\} \subseteq [3]$. The corresponding composition is $\mathbf{m} = (2, 2)$. Applying (6.8), we get

$$\alpha_3(\{1, 3\}) = \frac{2 \cdot 2}{24} 2! \mathcal{MLat}^2(\Delta_{1,4}, \Delta_{3,4}).$$

Furthermore, Theorem 2.16 provides a way to compute mixed valuations:

$$2! \mathcal{MLat}^2(\Delta_{1,4}, \Delta_{3,4}) = \text{Lat}^2(\Delta_{1,4} + \Delta_{3,4}) - \text{Lat}^2(\Delta_{1,4}) - \text{Lat}^2(\Delta_{3,4}).$$

By the comment after Theorem 2.14, for any polytope $\text{Lat}^r(P)$ is the coefficient of t^r in the Ehrhart polynomial $i(P, t)$. Hence, we can figure out the terms in the right hand side of the above equality by computing corresponding Ehrhart polynomials:

$$\begin{aligned} i(\Delta_{14} + \Delta_{34}, t) &= \frac{10}{3}t^3 + 5t^2 + \frac{11}{3}t + 1, \\ i(\Delta_{14}, t) &= \frac{1}{6}t^3 + t^2 + \frac{11}{6}t + 1, \\ i(\Delta_{34}, t) &= \frac{1}{6}t^3 + t^2 + \frac{11}{6}t + 1. \end{aligned}$$

Therefore,

$$2! \mathcal{MLat}^2(\Delta_{1,4}, \Delta_{3,4}) = 5 - 1 - 1 = 3,$$

and

$$\alpha_3(\{1, 3\}) = \frac{2 \cdot 2}{24} \cdot 3 = \frac{1}{2},$$

which agrees with (3.9) since $F_{\{1,3\}}$ is a facet.

Example 6.9. Consider $S = \{i\}$. Then F_S is an edge, and the corresponding composition \mathbf{m} consists of one copy of 2 and $n-1$ copies of 1. Hence,

$$\alpha_n(\{i\}) = \frac{2}{(n+1)!} \text{Lat}^1(\Delta_{i, n+1}).$$

We now use the above formula to compute $\alpha_5(\{i\})$ which are $\alpha(E, \Pi_5)$ for edges E of Π_5 . Again, $\text{Lat}^1(\Delta_{i,6})$ is the linear coefficient of the Ehrhart polynomial of $\Delta_{i,6}$. So we compute these polynomials:

$$\begin{aligned} i(\Delta_{1,6}, t) &= \frac{1}{120}t^5 + \frac{1}{8}t^4 + \frac{17}{24}t^3 + \frac{15}{8}t^2 + \frac{137}{60}t + 1, \\ i(\Delta_{2,6}, t) &= \frac{13}{60}t^5 + \frac{3}{2}t^4 + \frac{47}{12}t^3 + 5t^2 + \frac{101}{30}t + 1, \\ i(\Delta_{3,6}, t) &= \frac{11}{20}t^5 + \frac{11}{4}t^4 + \frac{23}{4}t^3 + \frac{25}{4}t^2 + \frac{37}{10}t + 1. \end{aligned}$$

Therefore,

$$\begin{aligned} \alpha_5(\{1\}) &= \frac{2}{6!} \text{Lat}^1(\Delta_{1,6}) = \frac{2}{720} \frac{137}{60} = \frac{137}{21600}, \\ \alpha_5(\{2\}) &= \frac{2}{6!} \text{Lat}^1(\Delta_{2,6}) = \frac{2}{720} \frac{101}{30} = \frac{101}{10800}, \\ \alpha_5(\{3\}) &= \frac{2}{6!} \text{Lat}^1(\Delta_{3,6}) = \frac{2}{720} \frac{37}{10} = \frac{37}{3600}, \end{aligned}$$

agreeing with Example 5.5.

7. TORIC PERSPECTIVE

McMullen's originally showed that there are infinitely many solutions to McMullen's formula. In principle, even if Berline-Vergne's construction is not positive on the braid arrangement fan \mathfrak{B}_n , there may be another construction that is positive. Our main purpose in this section is to prove Proposition 7.2, which says in particular that if any construction coming from Pommersheim-Thomas methods is positive on \mathfrak{B}_n , then Berline-Vergne's construction is also positive. For this we use a standard argument in toric varieties [8, Section 5.3].

Setup 7.1. Let M a lattice, $P \subset M \otimes_{\mathbb{Z}} \mathbb{R}$ be an n dimensional lattice polytope, Δ its normal fan, subdivided if necessary to obtain a unimodular fan, and $X = X(\Delta)$ the corresponding toric variety. If P has the facet description

$$P = \{\mathbf{x} \in \mathbb{R}^n : \langle \mathbf{x}, \mathbf{n}_{F_i} \rangle \geq -a_i\}$$

where F_i are the facets of P and \mathbf{n}_{F_i} are their normal vectors, then P determines a divisor

$$D_P = \sum_i a_i D_{F_i}$$

where D_{F_i} is the divisor associated to the facet F_i . The divisor D is Cartier and it is generated by its global sections.

Since the torus invariant cycles generate the Chow ring, the Todd class of X has a (non-unique) expression of the form

$$(7.1) \quad \text{Td}(X) = \sum_{\sigma \in \Delta} r_{\sigma} [V(\sigma)].$$

Such an expression gives a solution to McMullen's formula

$$(7.2) \quad \text{Lat}(P) = \sum_{F \subset P} \alpha(F, P) \text{nvol}(F)$$

with $\alpha(F, P) = r_\sigma$, where $\sigma = \text{ncone}(F, P)$. We briefly explain how.

First of all, we have the following two key facts about the divisor D_P :

- (1) $\dim H^0(X, D_P) = \text{Lat}(P)$.
- (2) $H^j(X, D_P) = 0$ for $j > 0$.

Combined together, we get $\chi(X, D_P) = \text{Lat}(P)$. Now we can use the Riemann-Roch-Hirzebruch theorem to compute the euler characteristic.

$$\chi(X, D_P) = \deg(\text{ch}(D_P) \cdot \text{Td}(X))_0.$$

In the present case, where D_P is a divisor, we have

$$\text{ch}(D_P) = \sum_{k=0}^n \frac{D_P^k}{k!}.$$

The zero sub index means that we only care about the zero dimensional part of the intersection, so we can write

$$(7.3) \quad \chi(X, D_P) = \sum_{k=0}^n \sum_{\dim \sigma = k} r_\sigma \deg \left(\frac{D_P^k}{k!} [V(\sigma)] \right).$$

One of the pleasant connections between toric varieties and discrete geometry is the relation

$$\deg \left(\frac{D_P^k}{k!} [V(\sigma)] \right) = \text{nvol}(F_\sigma),$$

where F_σ is the face of P with normal cone σ . Combined with $\chi(X, D_P) = \text{Lat}(P)$ we get a solution for McMullen's formula

$$\text{Lat}(P) = \sum_{F \subseteq P} \alpha(F, P) \text{nvol}(F)$$

with $\alpha(F, P) = r_\sigma$, where $\sigma = \text{ncone}(F, P)$.

Originally, Danilov asked if the coefficients in the Equation (7.1) could be given depending just on σ . Even though this is more general than McMullen's formula, Berline-Vergne's construction actually solve Danilov's question as well [3].

Proposition 7.2. *Let Δ be the braid arrangement fan \mathfrak{B}_n . The following are equivalent:*

- (1) *The Todd class is positive with respect to the torus invariant cycle, that is*

$$\text{Td}(X(\Delta)) = \sum_{\sigma \in \Delta} r_\sigma [V(\sigma)],$$

with $r_\sigma > 0$.

- (2) *The regular permutohedron Π_n is BV- α -positive.*

Note that the condition in part (1) of Proposition 7.2 is much weaker than Danilov's condition, since we only need to choose r_σ for σ in the braid arrangement fan Δ , and do not need to worry about assignments to cones in other fans or whether (1.1) holds for other fans.

Proof. Clearly we have (2) \Rightarrow (1) since Berline-Vergne's construction solves Danilov's question.

Now assuming (1), we will prove (2). By Corollary 6.6, we only need to show that the mixed lattice points valuation \mathcal{MLat}^k on hypersimplices is always positive. We modify the Riemann-Roch-Hirzebruch argument to compute this mixed valuations.

Consider $P_i = \Delta_{i,n+1}$ for $1 \leq i \leq n$. Each of them is compatible with the fan \mathfrak{B}_n , so they each define a divisor D_i . Clearly, the divisor $D_P := w_1 D_1 + \cdots + w_n D_n$ corresponds to the polytope $P = w_1 P_1 + \cdots + w_n P_n$. Now we use the Riemann-Roch-Hirzebruch argument as above. We have that:

$$(7.4) \quad \text{Lat}(P) = \chi(X, D_P) = \sum_{k=0}^n \sum_{\dim \sigma = k} r_\sigma \deg \left(\frac{D_P^k}{k!} [V(\sigma)] \right).$$

Since $D^k = (w_1 D_1 + \cdots + w_n D_n)^k$, it follows naturally that $\text{Lat}(P)$ is a polynomial in the w_i 's. Let $S = \{s_1, \dots, s_k\}$, and consider coefficient of squarefree monomials $w_S := \prod_{i \in S} w_i$ in this polynomial. On the one hand, it is clearly given by

$$\sum_{\dim \sigma = k} r_\sigma \deg \left(\frac{D_S}{k!} [V(\sigma)] \right),$$

where $D_S = \prod_{i \in S} D_i$. On the other hand, similar as in the proof of Theorem 6.5, this coefficient is given by (6.5). Therefore, we get

$$(7.5) \quad k! \mathcal{MLat}^k(\Delta_{s_1, n+1}, \Delta_{s_2, n+1}, \dots, \Delta_{s_k, n+1}) = \sum_{\dim \sigma = k} r_\sigma \deg \left(\frac{D_S}{k!} [V(\sigma)] \right),$$

where $D_S = \prod_{i \in S} D_i$. Since D_i are generated by global sections, each intersection product $D_S[V(\sigma)]$ of the right hand side of (7.5) has positive degree. Therefore we have that the value of (7.5) is nonnegative for $r_\sigma > 0$. It is left to show this value cannot be zero.

We assume to the contrary that the right hand side of (7.5) is equal to 0. Then

$$D_S[V(\sigma)] = 0 \quad \forall \sigma \text{ of codimension } k.$$

But the torus invariant cycles generate the Chow groups [8, Section 5.1]. so it follows that

$$(7.6) \quad D_S \cdot C = 0 \quad \forall C \in A_k(X),$$

in the Chow ring. Note that the coefficient of $w_{[n]} = w_1 w_2 \dots w_n$ of the polynomial (7.4) is given by $r_\Delta \deg(D_1 D_2 D_3 \cdots D_n)$ which is strictly positivity by Lemma 6.3. Hence, we have

$$\deg(D_1 D_2 D_3 \cdots D_n) = \deg(D_{[n]}) > 0.$$

So $D_{[n]}$ is nonzero in the Chow ring.

Now let \overline{S} be the set complement of S in $[n]$. We have that $D_{\overline{S}} \in A_k(X)$ and $D_S \cdot D_{\overline{S}} = D_{[n]} \neq 0$, contradicting Equation (7.6). \square

Proposition 7.2 provides us another way to attack Conjecture 1.3 using theory of toric varieties. For example, any construction that answers Danilov's question can potentially provide a way to prove our conjectures.

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